Statistical properties of Schrödinger real and imaginary cat states

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Abstract

We study the photon statistics in the superpositions of coherent states $|\alpha\rangle$ and $|\alpha^*\rangle$ named "Schrödinger real and imaginary cat states". The oscillatory character of the photon distribution function (PDF) emerging due to the quantum interference between the two components is shown, and the phenomenon of the quadrature squeezing is observed for the moderate values of $|\alpha| \sim 1$. Despite the quantity $\langle\triangle n^2\rangle/\langle n\rangle$ tends to the unit value (like in the Poissonian PDF) at $|\alpha| \gg 1$, the photon statistics is essentially non-Poissonian for all values of $|\alpha|$. The factorial moments and cumulants of the PDF are calculated, and the oscillations of their ratio are demonstrated.

1 Introduction

The coherent states $|\alpha\rangle$ introduced by Glauber [1] with respect to problems of quantum optics are eigenstates of the photon annihilation operator $\hat{a}$ ($\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$), their eigenvalues covering the entire complex plane. Since the Heisenberg uncertainty relation [2] is minimized in coherent states, the latter are interpreted as the "classical" ones. The photon distribution function (PDF) in the coherent state is the usual Poisson distribution, which is a smooth function with a single maximum near the mean photon number.

In nonclassical Gaussian states of light the PDFs possess a number of specific properties related to the effect of quadrature squeezing [3, 4]. The PDF for one-mode Gaussian states of electromagnetic field was discussed in [5]. In [6] the single-mode *correlated* state of light was introduced, which minimizes the Schrödinger - Robertson uncertainity relation [7, 8]. The oscillatory character of PDF for squeezed and correlated Gaussian light was discussed in [9, 10]. Different nonclassical states of light might play an essential role in nonlinear processes [11].

By using different finite or infinite sums of coherent states for a one-mode quantum harmonic oscillator various non-Gaussian nonclassical states of light may be constructed. Recently, there have been much interest in the properties and in the generation of so-called "Schrödinger cats" [12] in the context of quantum optics [13]-[15]. These quantum superpositions of the finite or infinite number of coherent states have various nonclassical characteristics emerging due to the quantum interference between summands.

The first example of such kind was constructed in [16], where the concept of the even and odd coherent states $|\alpha_\pm\rangle = N_\pm(|\alpha|)(|\alpha\rangle \pm |-\alpha\rangle)$ was introduced. These states satisfy the

equation $\hat{a}^2|\alpha_\pm\rangle = \alpha^2|\alpha_\pm\rangle$. Moreover, they manifest such remarkable nonclassical properties as oscillations in photon number distribution, sub-Poissonian statistics, and the quadrature squeezing. The methods of generating and detecting the even and odd states in quantum optics were discussed, e.g., in [17, 18].

The aim of the present article is to study the photon statistics in another representative of nonclassical states: the "real" (RCS) and "imaginary" (ICS) coherent states (or "real" and "imaginary cats").

$$|\widetilde{\alpha_\pm}\rangle = \widetilde{N_\pm}(\alpha,\alpha^*)\left(|\alpha\rangle \pm |\alpha^*\rangle\right), \tag{1}$$

the normalization constant being given by ($\alpha \equiv |\alpha|e^{i\varphi}$)

$$\widetilde{N_\pm}(\alpha,\alpha^*) = \frac{1}{\sqrt{2}}\left(1 \pm \mathrm{Re}\, e^{\alpha^2-|\alpha|^2}\right)^{-1/2} = \frac{1}{\sqrt{2}}\left(1 \pm e^{-2|\alpha|^2\sin^2\varphi}\cos(|\alpha|^2\sin 2\varphi)\right)^{-1/2}. \tag{2}$$

Similar superpositions, $|\alpha\rangle + |-\alpha^*\rangle$, were considered from the group-theoretical point of view in [19], where they were called "charged Schrödinger cats". All the states of this family belong to the class of superposition states discussed in [13] and [20]. The interference between $|\alpha\rangle$ and $|\alpha^*\rangle$ leads again to the non-Poissonian (modulated Poissonian) PDF. Evidently, the statistical properties of real and imaginary cat states, as well as the degree of squeezing, depend essentially on both the modulus $|\alpha|$ and the phase difference 2φ between α and α^*. It is interesting, however, that the correlation coefficient between the quadrature components identically equals zero.

We obtain the generating function $G(n)$ of factorial moments $F(n)$, and use it to find the analytical expressions for the photon means, dispersions, and Mandel's Q-parameter. Besides, we calculate the cumulants $K(n)$ of the PDF and demonstrate the oscillatory behaviour (essentially nonclassical) of the ratio $H(n) = F(n)/K(n)$.

The parameter $H(n)$ was shown in [21] to play an essential role in analysing the particle multiplicity distributions in high energy physics, since in experiments it demonstrates an oscillatory behaviour, which is sensitive to the detailes of the state of particles created in high energy collisions [22]. For the squeezed and correlated states of the electromagnetic field this parameter was discussed in [23]. In the present article we concentrate on studying the behaviour of $H(n)$ for the Schrödinger real and imaginary cat states.

2 Means and variances of quadratures

Using the well known expansion

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \tag{3}$$

it is easy to derive the corresponding expansions for RCS and ICS:

$$|\widetilde{\alpha_+}\rangle = 2\widetilde{N_+} e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^n}{\sqrt{n!}} \cos(n\varphi)|n\rangle, \tag{4}$$

$$|\widetilde{\alpha_-}\rangle = 2i\widetilde{N_-} e^{-\frac{1}{2}|\alpha|^2} \sum_{n=1}^{\infty} \frac{|\alpha|^n}{\sqrt{n!}} \sin(n\varphi)|n\rangle, \tag{5}$$

where $0 \leq \varphi < 2\pi$, and $|n\rangle$ is the vector of the Fock space. The average values of the ladder operators $\hat{a}$, $\hat{a}^\dagger$ in real and imaginary cat states are invariant with respect to the Hermitian conjugation:

$$\langle \hat{a} \rangle_\pm = \langle \hat{a}^\dagger \rangle_\pm = 2|\widetilde{N_\pm}|^2 \left(\mathrm{Re}\, \alpha \pm \mathrm{Re} \left(\alpha e^{\alpha^2 - |\alpha|^2} \right) \right). \tag{6}$$

Consequently, the quadrature operators $\hat{p}$, $\hat{q}$ (we assume $\hbar = 1$)

$$\hat{p} = \frac{\hat{a} - \hat{a}^\dagger}{i\sqrt{2}}, \qquad \hat{q} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}} \tag{7}$$

possess the following mean values:

$$\langle q \rangle_\pm = \sqrt{2}\langle \hat{a}_\pm \rangle, \qquad \langle p \rangle_\pm = 0. \tag{8}$$

For the elements of the quadrature covariance matrix we get

$$\sigma_{qq}^{(\pm)} = \frac{1}{2} - 8|\widetilde{N_\pm}|^4 (\operatorname{Im}\alpha)^2 \left(|e^{\alpha^2 - |\alpha|^2}|^2 \pm \operatorname{Re} e^{\alpha^2 - |\alpha|^2} \right), \tag{9}$$

$$\sigma_{pp}^{(\pm)} = \frac{1}{2} + 4|\widetilde{N_\pm}|^2 (\operatorname{Im}\alpha)^2, \tag{10}$$

$$\sigma_{pq}^{(\pm)} \equiv \frac{1}{2}\langle \hat{p}\hat{q} + \hat{q}\hat{p} \rangle_\pm - \langle \hat{p} \rangle_\pm \langle \hat{q} \rangle_\pm = 0. \tag{11}$$

The last equality shows that both real and imaginary Schrödinger cat states have no correlation between the quadratures. Being rewritten in terms of $|\alpha|$ and φ, the quadrature variances read

$$\sigma_{qq}^{(\pm)} = \frac{1}{2} - 8\left(|\alpha \widetilde{N_\pm}^2| \sin\varphi \, e^{-|\alpha|^2 \sin^2\varphi} \right)^2 \left(e^{-2|\alpha|^2 \sin^2\varphi} \pm \cos(|\alpha|^2 \sin(2\varphi)) \right), \tag{12}$$

$$\sigma_{pp}^{(\pm)} = \frac{1}{2} + 4\left(|\alpha \widetilde{N_\pm}| \sin\varphi \right)^2. \tag{13}$$

The variance of the quadrature $\hat{p}$ always exceeds $\frac{1}{2}$. As to the quadrature $\hat{q}$, it is squeezed provided the inequality

$$\exp(-2|\alpha|^2 \sin^2\varphi) \pm \cos(|\alpha|^2 \sin 2\varphi) > 0$$

holds. Assuming $|\alpha|^2$ to be small, one can expand the summands in powers of $|\alpha|^2$. For the real cat state the limit of the inequality is $2 + \mathcal{O}\left(|\alpha|^2\right) > 0$, for any phase φ. The analogous limit for the imaginary cat state is less than zero for arbitrary phase. Thus, there is no quadrature squeezing in the ICS when $|\alpha|^2$ tends to zero. On the contrary, the squeezing in RCS exists for any φ up to $|\alpha|^2 = \pi/2$.

3 Photon distribution function, moments and cumulants

Using expansions (4) and (5) one gets the photon distribution functions

$$P_{\pm}(n) \equiv |\langle n|\widetilde{\alpha_{\pm}}\rangle|^2 = 2|\widetilde{N_{\pm}}(|\alpha|^2,\varphi)|^2 e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}(1 \pm \cos(2n\varphi)). \tag{14}$$

It is obvious that $P_{\pm}(n)$, being a modulated Poissonian distribution, is the sum of three Poissonian distributions with complex means, which became the usual Poissonian distribution with $\langle n\rangle = \alpha^2$ at real α. Oscillations of PDF are demonstrated in Figs. 1 and 2 both for RCS (a) and ICS (b) in the representative cases of $|\alpha|^2 = 1.5$ (Fig. 1) and $|\alpha|^2 = 9.0$ (Fig. 2), $\varphi = 0.3\pi$.

The simplest way to derive the statistical characteristics of the field (in particular, $\langle n\rangle$ and $\langle \triangle n^2\rangle \equiv \langle n^2\rangle - \langle n\rangle^2$) is to differentiate the generating function

$$G_{\pm}(\lambda) \equiv \sum_{n=0}^{\infty} P_{\pm}(n)\lambda^n = 2|\widetilde{N_{\pm}}|^2 e^{-|\alpha|^2}\left(e^{|\alpha|^2\lambda} \pm \mathrm{Re}\, e^{\alpha^2\lambda}\right) \tag{15}$$

with respect to the auxiliary real parameter λ. Then the factorial moments read

$$F(n) \equiv \left.\frac{d^n G}{d\lambda^n}\right|_{\lambda=1} = 2|\widetilde{N_{\pm}}(|\alpha|^2,\varphi)|^2|\alpha|^{2n}\left(1 \pm e^{-2|\alpha|^2\sin^2\varphi}\cos(|\alpha|^2\sin 2\varphi + 2n\varphi)\right). \tag{16}$$

The factorial moments up to the 10-th order are plotted in Fig. 3 for RCS (a) at $|\alpha|^2 = 1.5$ and $\varphi = 0.3\pi$. The plot of $F(n)$ for ICS at the same α seems similar to Fig. 3.

It is clear that in the case of $|\alpha|^2\sin^2\varphi \gg 1$ expression (16) tends to the classical (Poissonian) limit $|\alpha|^{2n}$. Nevertheless, the PDFs in both RCS and ICS dramatically differ from the Poissonian distribution for all values of $|\alpha|$. This means, in particular, that the knowledge of a few of the lowest factorial moments is not sufficient to decide whether the statistics is

Poissonian or not. In the limit $|\alpha|^2 \gg 1$ the difference becomes clear only for the moments having the orders not less than $n' \approx |\alpha|^2 \sin^2\varphi / \ln|\alpha|$. In the case of $|\alpha|^2 = 9.0$ and $\varphi = 0.3\pi$ we have $n' \approx 6$.

The mean photon number equals

$$\langle n\rangle_\pm \equiv \left.\frac{dG_\pm}{d\lambda}\right|_{\lambda=1} = 2|\alpha|^2 |\widetilde{N_\pm}(|\alpha|^2,\varphi)|^2 \left(1 \pm e^{-2|\alpha|^2\sin^2\varphi}\cos(|\alpha|^2\sin 2\varphi + 2\varphi)\right), \tag{17}$$

whereas Mandel's Q-parameter

$$Q_\pm \equiv \frac{\langle \triangle n^2\rangle_\pm - \langle n\rangle_\pm}{\langle n\rangle_\pm}$$

is equal to

$$Q_\pm = -\frac{|2\alpha\widetilde{N_\pm}|^4}{\langle n\rangle_\pm} e^{-2|\alpha|^2\sin^2\varphi}\sin^2\varphi\left(\cos^2\varphi\, e^{-2|\alpha|^2\sin^2\varphi} \pm \cos(|\alpha|^2\sin 2\varphi + 2\varphi)\right). \tag{18}$$

However, in the case under study this parameter is not suitable for the analysis of the photon statistics, since for large values of $|\alpha\sin\varphi|$ it is close to zero, although statistics remains essentially non-Poissonian. Actually the equality $Q \approx 0$ is only necessary, but not sufficient condition of the Poissonian statistics.

More informative characteristics of the photon statistics are the cumulants

$$K(n) \equiv \left.\frac{d^n \ln G}{d\lambda^n}\right|_{\lambda=1}. \tag{19}$$

To find the relation between the moments and cumulants note that

$$\frac{d}{d\lambda}G = G\frac{d\ln G}{d\lambda}.$$

Thus,

$$\frac{d^n}{d\lambda^n}G = \frac{d^{n-1}}{d\lambda^{n-1}}\left(G\frac{d\ln G}{d\lambda}\right) = \sum_{k=0}^{n-1}\begin{pmatrix} k \\ n-1\end{pmatrix}\frac{d^{k+1}\ln G}{d\lambda^{k+1}}\frac{d^{n-1-k}G}{d\lambda^{n-1-k}}. \tag{20}$$

Putting $\lambda = 1$ in (20) one gets the recursive relation

$$F(n) = \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} K(k) F(n-k), \tag{21}$$

followed by another one, which is useful for the numerical calculation of the cumulant of the n-th order through the cumulants of preceding orders and moments up to the n-th order,

$$K(n) = F(n) - \sum_{k=1}^{n-1} \frac{(n-1)!}{(k-1)!(n-k)!} K(k) F(n-k). \tag{22}$$

In particular,

$$\begin{aligned}
K_1 &= F_1, \\
K_2 &= F_2 - F_1^2, \\
K_3 &= F_3 - 3F_2F_1 + 2F_1^3, \\
K_4 &= F_4 - 4F_3F_1 - 3F_2^2 + 12F_1^2F_2 - 6F_1^4, \\
K_5 &= F_5 - 10F_2F_3 - 5F_1F_4 + 20F_1^2F_3 + 30F_2^2F_1 - 60F_1^3F_2 + 24F_1^5, \\
K_6 &= F_6 - 6F_1F_5 - 15F_2F_4 + 120F_1F_2F_3 + 30F_1^2F_4 - 120F_1^3F_3 \\
&\quad -270F_2^2F_1^2 + 360F_1^4F_2 - 10F_3^2 + 30F_2^3 - 120F_1^6.
\end{aligned}$$

The cumulants of the photon number up to $n = 10$ corresponding to $|\widetilde{\alpha_+}\rangle$-state with $|\alpha|^2 = 1.5$, $\varphi = 0.3\pi$ are plotted in Fig. 4 (a) to demonstrate the fast (factorial-like) divergence of a series of $K(n)$. Since the cat states with $|\alpha|^2 \sim 1$ are inherently nonclassical, the cumulants differ from zero even for small values of n. Their oscillating behavior is demonstrated in Fig. 4 (b).

The ratio $H(n) = F(n)/K(n)$ may be used instead of Q-parameter to evaluate how "nonclassical" is a quantum state [23]. Its oscillatory behavior in RCS and ICS with $|\alpha|^2 = 1.5$ and $|\alpha|^2 = 9.0$, $\varphi = 0.3\pi$ is seen from Fig. 5. The strong decrease at higher n is also seen: the factorial-like divergence of $K(n)$ surpasses the quasiexponential growth of $F(n)$.

The high amplitude oscillations of $H(n) = F(n)/K(n)$ for cat states with $|\alpha|^2 = 9.0$ admit a simple explanation : while $F(n)$ equals approximately the classical value $|\alpha|^{2n}$, the cumulants $K(n) \approx \pm 0$. Nevertheless, for large n the ratio $F(n)/K(n)$ damps even for $|\alpha|^2 = 9.0$. Note that the plot of $H(n)$ for the imaginary cat state with $|\alpha|^2 = 9.0$ is symmetrical to the analogous plot for the real cat state with respect to n-axis.

4 Conclusion

We have shown that both real and imaginary Schrödinger cat states manifest quadrature squeezing and non-Poissonian statistics. The factorial moments of the photon distribution function have proven to be monotonically growing, while the oscillatory factorial divergence to infinity is observed for the cumulants $K(n)$ when $n \gg 1$. The ratio $F(n)/K(n)$ has been found to be strongly oscillating for small n and damping when n grows. These properties confirm the nonclassical nature of the Schrödinger real and imaginary cat states.

If one considers the time evolution of quantum states in the case of a harmonic oscillator with unit frequency, then RCS and ICS will be transformed as

$$|\widetilde{\alpha_\pm, t}\rangle = \widetilde{N_\pm}(\alpha,\alpha^*)\left(|\alpha e^{-it}\rangle \pm |\alpha^* e^{-it}\rangle\right)$$

(with the same complex exponentials in both terms). Thus in the time-dependent case the complex parameters of the coherent states determining the superpositions do not preserve

the property to be mutually complex conjugated. However, the phase difference 2φ between these parameters is time invariant. Since the PDF is also time invariant, it is convenient and natural to choose the initial moment in such a way that the centers of the two coherent states wave packets would be located symmetrically with respect to the real axis. Just this natural choice has been made in the paper.

The multimode generalization of the real and imaginary cat states may be constructed analogously to the multimode generalization of the even and odd coherent states done in [24].

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Figure captions

Fig. 1. Photon distribution functions $P_+(n)$ for $|\widetilde{\alpha_+}\rangle$ (a) and $P_-(n)$ for $|\widetilde{\alpha_-}\rangle$ (b) with $|\alpha|^2 = 1.5$, $\varphi = 0.3\pi$.

Fig. 2. $P_+(n)$ (a) and $P_-(n)$ (b) for $|\alpha|^2 = 9.0$, $\varphi = 0.3\pi$.

Fig. 3. Factorial moments $F(n)$ of the photon number for $|\widetilde{\alpha_+}\rangle$ (a) and $|\widetilde{\alpha_-}\rangle$ (b) with $|\alpha|^2 = 1.5$, $\varphi = 0.3\pi$.

Fig. 4. Cumulants $K(n)$ of the photon number for $|\widetilde{\alpha_+}\rangle$-state up to the 10-th order (a) and up to the 5-th order (b) with $|\alpha|^2 = 1.5$, $\varphi = 0.3\pi$.

Fig. 5. Ratio $F(n)/K(n)$ for $|\widetilde{\alpha_+}\rangle$ (a,c) and $|\widetilde{\alpha_-}\rangle$ (b,d) with $|\alpha|^2 = 1.5$ (a,b), $|\alpha|^2 = 9.0$ (c,d) and $\varphi = 0.3\pi$.

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